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Weighted complexities of graph products and bundles

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Abstract

The complexity $\kappa(G)$ of a graph G is the number of spanning trees in G. In spite of its importance, most known methods for computing $\kappa(G)$ commonly have computational difficulties since they require to compute determinants or eigenvalues of matrices of the size of the order of a graph. In particular, they are not feasible for large graphs. However, many of them can be represented by some graph operations. A graph bundle is a notion containing a cartesian product of graphs and a (regular or irregular) graph covering. For a regular graph covering, H. Mizuno and I. Sato [Zeta functions for images of graph coverings by some operations, Interdiscip. Inform. Sci. 7 (2001) 53–60] computed its complexity. We extend their work to a graph bundle by deriving a factorized formula for the complexity: If a graph bundle has a regular fibre, its complexity can be factorized into the complexity of the base graph and determinants of smaller-size matrices. For the complexities of the cartesian products of graphs, several computing formulae are already known. However, they also used somewhat complicated calculations of determinants, eigenvalues or trigonometric equations. We reduce such complication for the known cases of the ladder, the Möbius ladder and the prism, by simply deriving the factorized formulae for their complexities. New concrete formulae for the complexities of the product $P_n \times K_m$ of the path P_n and the complete graph K_m and those of K_m-bundles over the cycle C_n are also derived as generalizations of the prism and the Möbius ladder.

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1. Introduction

Graphs in this paper are finite and simple (with no loops and no multiple edges). Let G be a connected (undirected) graph with vertex set $V(G)$ and edge set $E(G)$, and let ν_G and ε_G denote the numbers of vertices and edges of G, respectively. The *complexity* $\kappa(G)$ of G is the number of spanning trees in G. The complexity of a graph is an interesting, well studied combinatorial object (see [2,5,7,26]) and is also of interest in several applications [3,14,26]. The literature addressing the problem of the complexity of a given graph is very extensive, and a number of different computational methods are known (see [1,2,4,7,8,15,22,24,27]). The classic result [16] known as the *matrix tree theorem*, which is attributed to Kirchhoff, goes way back to 1847. Let $V(G) = \{v_1, \ldots, v_{\nu_G}\}$. The *adjacency matrix* $\mathbf{A} = \mathbf{A}_G = (a_{ij})$ of G is the $\nu_G \times \nu_G$ matrix with $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. Let $\mathbf{D} = \mathbf{D}_G$ denote the diagonal matrix whose (i,i)-entry is the degree $\deg_G(v_i)$ of v_i. The matrix tree theorem states that any cofactor of the *Laplacian matrix* $\mathbf{L} = \mathbf{L}_G = \mathbf{D} - \mathbf{A}$ is equal to $\kappa(G)$. Another algebraic description for $\kappa(G)$ is due to Temperley [27]:

$$\kappa(G) = \frac{1}{\nu_G^2} \det(\mathbf{L} + \mathbf{J}),$$

where $\mathbf{J}$ is the $\nu_G \times \nu_G$ matrix whose entries are all 1. Let $\mu_1 (=0) \le \mu_2 \le \cdots \le \mu_{\nu_G}$ denote the eigenvalues of the Laplacian matrix $\mathbf{L}$. Kel'mans and Chelnokov [15] have shown that

$$\kappa(G) = \frac{1}{\nu_G} \prod_{j=2}^{\nu_G} \mu_j.$$

Recently, Northshield [22] introduced a new method for computing the complexity via the study of the Ihara function $Z_G(u)$, $u \in \mathbb{C}$:

$$Z_G^{-1}(u) = (1-u^2)^{\varepsilon_G - \nu_G} \det(\mathbf{I}_{\nu_G} - u\mathbf{A} + u^2\mathbf{Q}),$$

where $\mathbf{I}_{\nu_G}$ is the identity matrix of order ν_G and $\mathbf{Q} = \mathbf{D} - \mathbf{I}_{\nu_G}$. In this paper, we call the matrix

$$\mathbf{\Phi}_G(u) = \mathbf{I}_{\nu_G} - u\mathbf{A} + u^2\mathbf{Q},$$

the *generalized Laplacian matrix* of G. Note that $\mathbf{\Phi}_G(1)$ is the Laplacian matrix $\mathbf{L}_G$ of G. Northshield computed $\kappa(G)$ in terms of the generalized Laplacian matrix of G as follows.

Theorem 1 (*Northshield*). *For a connected graph G,*

$$\kappa(G) = \frac{1}{2(\varepsilon_G - \nu_G)} \delta'_G(1),$$

where $\delta_G(u) = \det \mathbf{\Phi}_G(u)$.

The methods described give general solutions for computing $\kappa(G)$. However, their common computational difficulties are the computing of determinants or eigenvalues of $\nu_G \times \nu_G$ matrices. In particular, they are not feasible for large graphs. However, many large graphs can be represented in the form of a product of more elementary graphs or a covering of smaller graphs. A graph bundle is a notion containing a cartesian product of graphs and a covering of a graph. We aim, in this paper, to give a simple method for computing the complexity of a graph bundle.

We show that the determinant of the generalized Laplacian matrix $\mathbf{\Phi}_G(u)$ can be factorized into determinants of smaller-size matrices when G is a graph bundle with regular fibre. Using this, we obtain a formula for the complexity of a graph bundle in terms of those of the base graph and determinant-factors. Some graph bundles have base graphs whose complexities are already known and determinant-factors which can be calculated precisely. Some of them will be illustrated in the last section. Actually, we carry out our research into the complexity of a weighted graph which is a general concept of a graph.

A *weight function* of G is a complex-valued function $\omega : E(G) \to \mathbb{C}\backslash\{0\}$ and $\omega(e)$ is called the *weight* of an edge $e = v_i v_j \in E(G)$. In this case, a pair (G, ω) is called a *weighted graph*. The (*weighted*) *adjacency matrix* $\mathbf{A}_{(G,\omega)} = (\omega_{ij})$ of a weighted graph (G, ω) is the $\nu_G \times \nu_G$ matrix such that

$$\omega_{ij} = \begin{cases} \omega(v_i v_j) & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the weighted adjacency matrix $\mathbf{A}_{(G,\omega)}$ is symmetric. Let $\mathbf{D}_{(G,\omega)} = (d_{ij})$ be the $\nu_G \times \nu_G$ diagonal matrix with $d_{ii} = \sum_{j \neq i} \omega(v_i v_j)$ and $\mathbf{Q}_{(G,\omega)} = \mathbf{D}_{(G,\omega)} - \mathbf{I}_{\nu_G}$. The number $d_{ii} = \sum_{j \neq i} \omega(v_i v_j)$ is called the *weight* of the vertex v_i and denoted by $\omega(v_i)$. The (*weighted*) *Laplacian matrix* $\mathbf{L}_{(G,\omega)}$ of a weighted graph (G, ω) is defined by $\mathbf{L}_{(G,\omega)} = \mathbf{D}_{(G,\omega)} - \mathbf{A}_{(G,\omega)}$. Also, the *generalized Laplacian matrix* of a weighted graph (G, ω) is defined by

$$\mathbf{\Phi}_{(G,\omega)}(u) = \mathbf{I}_{\nu_G} - u\mathbf{A}_{(G,\omega)} + u^2 \mathbf{Q}_{(G,\omega)}.$$

Clearly, $\mathbf{\Phi}_{(G,\omega)}(1)$ is the Laplacian matrix $\mathbf{L}_{(G,\omega)}$ of a weighted graph (G, ω). For a subgraph H of G, write

$$\omega_\times(H) = \prod_{e \in E(H)} \omega(e), \qquad \omega_+(H) = \sum_{e \in E(H)} \omega(e).$$

The number

$$\kappa(G, \omega) = \sum_T \omega_\times(T),$$

where the sum runs over all spanning trees T in G is called the *weighted complexity* of a weighted graph (G, ω). If $\omega(e) = 1$ for every $e \in E(G)$, $\kappa(G, \omega)$ is just the complexity $\kappa(G)$ in the usual sense. Chaiken [6] showed that any cofactor of the weighted Laplacian matrix $\mathbf{L}_{(G,\omega)}$ is equal to $\kappa(G, \omega)$.

Mizuno and Sato [20] generalized Theorem 1 to a weighted graph as follows.

Theorem 2 (*Mizuno and Sato*). *Let G be a connected graph with ν_G vertices and let ω be a weight function of G. Then*

$$\kappa(G, \omega) = \frac{1}{2(\omega_+(G) - \nu_G)} \delta'_{(G,\omega)}(1),$$

where $\delta_{(G,\omega)}(u) = \det \mathbf{\Phi}_{(G,\omega)}(u)$.

In this paper, we study the weighted complexity of a graph bundle which is a generalization of a graph covering and a cartesian product of two graphs. In Section 2, we examine the generalized Laplacian matrix of a graph bundle in order to get its decomposition formula. In Section 3, by using the decomposition formula, we derive a formula for the weighted complexity of a graph bundle with regular fibre. By using this, we compute, in Sections 4 and 5, the weighted complexities of a (regular or irregular) graph covering and a cartesian product of two graphs when at least one of them is regular. In Section 6, we present some examples to show the efficiency of our formulae.

For a general theory of group representations and graph coverings, the reader is referred to [25] and [12], respectively.

2. Generalized Laplacian matrices of graph bundles

Let G be a connected graph and let Γ be a finite group. Denote the arc set of G by $A(G)$. A *Γ-voltage assignment* of G is a function $\phi : A(G) \to \Gamma$ such that $\phi(a^{-1}) = \phi(a)^{-1}$ for every $a \in A(G)$. We denote the set of Γ-voltage assignments of G by $C^1(G; \Gamma)$.

Let F be another graph and let $\mathrm{Aut}(F)$ denote the automorphism group of F. Let $\phi \in C^1(G; \mathrm{Aut}(F))$. Now, we construct a new graph $G \times^{\phi} F$ as follows:

(1) $V(G \times^{\phi} F) = V(G) \times V(F)$;

(2) $(u_1, v_1)(u_2, v_2) \in E(G \times^{\phi} F)$ if either

  (i) $u_1 u_2 \in E(G)$ and $v_1^{\phi(u_1,u_2)} = v_2$ (called a *base-type edge*) or

  (ii) $u_1 = u_2$ and $v_1 v_2 \in E(F)$ (called a *fibre-type edge*).

The graph $G \times^{\phi} F$ is called the *F-bundle* over G associated with ϕ (or simply a *graph bundle*) and the graphs G and F are called the *base* and the *fibre* of the graph bundle $G \times^{\phi} F$, respectively. The first coordinate projection induces the bundle projection $p^{\phi} : G \times^{\phi} F \to G$. Note that the map p^{ϕ} maps vertices to vertices, base-type edges to edges and fibre-type edges to vertices. If $F = \overline{K}_m$, the complement of the complete graph K_m with m vertices, then an F-bundle over G is just an m-fold graph covering of G. If $\phi(a)$ is the identity in $\mathrm{Aut}(F)$ for all $a \in A(G)$, then $G \times^{\phi} F$ is just the cartesian product of G and F (for more on bundles, see [9,17,18,21,23]).

If G and F are weighted graphs with weight functions ω_G and ω_F, respectively, one can define a weight function $\widetilde{\omega} : E(G \times^{\phi} F) \to \mathbb{C}\backslash\{0\}$ of the graph bundle $G \times^{\phi} F$ in a natural way:

$$\widetilde{\omega}((u_i, v_k)(u_j, v_\ell)) = \begin{cases} \omega_G(u_i u_j) & \text{if } u_i u_j \in E(G) \text{ and } v_k^{\phi(u_i,u_j)} = v_\ell, \\ \omega_F(v_k v_\ell) & \text{if } u_i = u_j \text{ and } v_k v_\ell \in E(F). \end{cases}$$

Note that the weights of base-type edges are inherited from the weights of edges of the base graph G, while the weights of fibre-type edges are the weights of edges of the fibre graph F.

Now we express the generalized Laplacian matrix of a weighted graph bundle $(G \times^{\phi} F, \widetilde{\omega})$ as a direct sum of block matrices. To do this, we first examine the weighted adjacency matrix $\mathbf{A}_{(G \times^{\phi} F, \widetilde{\omega})}$. Let $V(G) = \{u_1, u_2, \ldots, u_{\nu_G}\}$ and

$V(F) = \{v_1, v_2, \ldots, v_{\nu_F}\}$. We define an ordering $<$ on $V(G \times^\phi F)$ as follows: For $(u_i, v_k), (u_j, v_\ell) \in V(G \times^\phi F)$, $(u_i, v_k) < (u_j, v_\ell)$ if either $k < \ell$ or $k = \ell$ and $i < j$.

Let $\vec{G}$ denote the digraph obtained from G by replacing each edge $u_i u_j = \{u_i, u_j\}$ with the two opposing arcs (u_i, u_j) and (u_j, u_i). For each $\sigma \in \mathrm{Aut}(F)$, let $\vec{G}_{(\phi,\sigma)}$ denote the subdigraph of the digraph $\vec{G}$ whose arc set is $\phi^{-1}(\sigma)$. Then the digraph $\vec{G}$ is the arc-disjoint union of subdigraphs $\vec{G}_{(\phi,\sigma)}$, $\sigma \in \mathrm{Aut}(F)$. For simplicity, we write $\mathbf{A}(\sigma) = \mathbf{A}_{(\vec{G}_{(\phi,\sigma)},\omega_G)}$ whose (u, v)-entry is $w_G(u, v)$ if $\phi(u, v) = \sigma$ and is zero otherwise. Then $\mathbf{A}_{(G,\omega_G)} = \sum_{\sigma \in \mathrm{Aut}(F)} \mathbf{A}(\sigma)$. Let S_F denote the symmetric group on $V(F)$. Let $\mathbf{P}(\sigma)$ denote the $\nu_F \times \nu_F$ permutation matrix associated with $\sigma \in S_F$ corresponding to the action of S_F on $V(F)$, i.e.,

$$\mathbf{P}(\sigma)_{k\ell} = \begin{cases} 1 & \text{if } v_k^\sigma = v_\ell, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Then for any $\sigma, \tau \in S_F$, $\mathbf{P}(\sigma\tau) = \mathbf{P}(\sigma)\mathbf{P}(\tau)$.

The tensor product $\mathbf{X} \otimes \mathbf{Y}$ of the matrices $\mathbf{X}$ and $\mathbf{Y}$ is considered as the matrix $\mathbf{Y} = (y_{k\ell})$ having the entry $y_{k\ell}$ replaced by the matrix $y_{k\ell}\mathbf{X}$. A computation similar to the one in Theorem 1 in [18] gives

$$\mathbf{A}_{(G \times^\phi F, \widetilde{\omega})} = \left(\sum_{\sigma \in \Gamma} \mathbf{A}(\sigma) \otimes \mathbf{P}(\sigma) \right) + \mathbf{I}_{\nu_G} \otimes \mathbf{A}_{(F,\omega_F)},$$

where Γ is the subgroup of $\mathrm{Aut}(F)$ generated by $\{\phi(a) \mid a \in A(G)\}$.

Next we compute the matrix $\mathbf{Q}_{(G \times^\phi F, \widetilde{\omega})}$. For a vertex $(u_i, v_k) \in V(G \times^\phi F)$, its weight is $\omega_G(u_i) + \omega_F(v_k)$. Thus

$$\mathbf{D}_{(G \times^\phi F, \widetilde{\omega})} = \mathbf{D}_{(G,\omega_G)} \otimes \mathbf{I}_{\nu_F} + \mathbf{I}_{\nu_G} \otimes \mathbf{D}_{(F,\omega_F)},$$

and hence

$$\begin{aligned} \mathbf{Q}_{(G \times^\phi F, \widetilde{\omega})} &= \mathbf{D}_{(G \times^\phi F, \widetilde{\omega})} - \mathbf{I}_{\nu_G \nu_F} = \mathbf{D}_{(G,\omega_G)} \otimes \mathbf{I}_{\nu_F} + \mathbf{I}_{\nu_G} \otimes \mathbf{D}_{(F,\omega_F)} - \mathbf{I}_{\nu_G} \otimes \mathbf{I}_{\nu_F} \\ &= (\mathbf{D}_{(G,\omega_G)} - \mathbf{I}_{\nu_G}) \otimes \mathbf{I}_{\nu_F} + \mathbf{I}_{\nu_G} \otimes \mathbf{D}_{(F,\omega_F)} \\ &= \mathbf{Q}_{(G,\omega_G)} \otimes \mathbf{I}_{\nu_F} + \mathbf{I}_{\nu_G} \otimes \mathbf{D}_{(F,\omega_F)}. \end{aligned}$$

Now, the following theorem follows immediately.

Theorem 3. *Let (G, ω_G) and (F, ω_F) be weighted graphs and let ϕ be an* Aut(F)-*voltage assignment. Then the generalized Laplacian matrix of a weighted graph bundle $(G \times^\phi F, \widetilde{\omega})$ is*

$$\begin{aligned} \mathbf{\Phi}_{(G \times^\phi F, \widetilde{\omega})}(u) = \mathbf{I}_{\nu_G \nu_F} &- u \left(\sum_{\sigma \in \Gamma} \mathbf{A}(\sigma) \otimes \mathbf{P}(\sigma) + \mathbf{I}_{\nu_G} \otimes \mathbf{A}_{(F,\omega_F)} \right) \\ &+ u^2 \left(\mathbf{Q}_{(G,\omega_G)} \otimes \mathbf{I}_{\nu_F} + \mathbf{I}_{\nu_G} \otimes \mathbf{D}_{(F,\omega_F)} \right), \end{aligned}$$

where Γ is the subgroup of Aut(F) *generated by* $\{\phi(a) \mid a \in A(G)\}$. □

If $\omega_G(u_i u_j) = \omega_F(v_k v_\ell) = 1$ for all $u_i u_j \in E(G)$ and for all $v_k v_\ell \in E(F)$, then we have an unweighted version.

Corollary 4. *Let G and F be graphs and let ϕ be an* $\mathrm{Aut}(F)$*-voltage assignment. Then the generalized Laplacian matrix of a graph bundle $G \times^{\phi} F$ is*

$$\begin{aligned}\mathbf{\Phi}_{G\times^{\phi}F}(u) = \mathbf{I}_{\nu_G\nu_F} - u\left(\sum_{\sigma\in\Gamma}\mathbf{A}(\sigma)\otimes\mathbf{P}(\sigma) + \mathbf{I}_{\nu_G}\otimes\mathbf{A}_F\right)\\ + u^2\left(\mathbf{Q}_G\otimes\mathbf{I}_{\nu_F} + \mathbf{I}_{\nu_G}\otimes\mathbf{D}_F\right),\end{aligned}$$

where Γ is the subgroup of $\mathrm{Aut}(F)$ *generated by $\{\phi(a) \mid a \in A(G)\}$.* □

In the generalized Laplacian matrix given in Theorem 3, the matrix $\mathbf{A}(\sigma)$ may not be diagonalizable. Though the second factors in the summands of the generalized Laplacian matrix, which are $\mathbf{I}_{\nu_F}$, $\mathbf{P}(\sigma)$, $\mathbf{A}_{(F,\omega_F)}$, $\mathbf{D}_{(F,\omega_F)}$, are diagonalizable, they may not commute with each other, that is, $\mathbf{A}_{(F,\omega_F)}\mathbf{D}_{(F,\omega_F)} \neq \mathbf{D}_{(F,\omega_F)}\mathbf{A}_{(F,\omega_F)}$ in general. To find a condition under which all of the second factors are simultaneously factorizable, we need the following definition.

A weighted graph (F, ω_F) is said to be (r, c)-*regular* if the underlying graph F is regular of valency r and $\omega_F(v) = c$ for all $v \in V(F)$. Let us *assume* that the fibre (F, ω_F) is (r, c)-regular from now on. Then the generalized Laplacian matrices in Theorem 3 and Corollary 4 are similar to direct sums of smaller-size matrices. To show this, first note that each row and column sum of the adjacency matrix $\mathbf{A}_{(F,\omega_F)}$ of (F, ω_F) is equal to c when (F, ω_F) is (r, c)-regular. Such a matrix is known as a *generalized doubly stochastic matrix* corresponding to c, that is, it is a square matrix over $\mathbb{C}$ such that each row and column sum is equal to c (see [10]). Gibson [10] gave the following characterization.

Theorem 5 (*Gibson*). *An $n \times n$ matrix $\mathbf{A}$ over $\mathbb{C}$ is a generalized doubly stochastic matrix corresponding to a complex number c if and only if there exist permutation matrices $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_t$ and complex numbers $c_1, c_2, \ldots, c_t$ such that $\sum_{i=1}^{t} c_i = c$ and*

$$\mathbf{A} = c_1\mathbf{P}_1 + c_2\mathbf{P}_2 + \cdots + c_t\mathbf{P}_t.$$

In fact, for any nonzero $n \times n$ generalized doubly stochastic matrix $\mathbf{A}$, one can choose n nonzero entries so that no two of them are on the same line. Let $\mathbf{P}_1$ be the permutation matrix having entry 1's in the same positions as those occupied by the chosen n nonzero entries of $\mathbf{A}$, and let c_1 be the number having the smallest absolute value among these n nonzero entries. Then $\mathbf{A}_1 = \mathbf{A} - c_1\mathbf{P}_1$ is also a generalized doubly stochastic matrix corresponding to $c - c_1$, and at least one more 0 appears in $\mathbf{A}_1$ than in $\mathbf{A}$. Hence we may iterate the argument on $\mathbf{A}_1$ and eventually obtain the desired linear decomposition of $\mathbf{A}$ with $\sum_{i=1}^{t} c_i = c$.

In particular, the adjacency matrix $\mathbf{A}_{(F,\omega_F)}$ of a (r, c)-regular graph (F, ω_F) can be written as a linear combination of permutation matrices, say

$$\mathbf{A}_{(F,\omega_F)} = \sum_{i=1}^{t} c_i\mathbf{P}(\sigma_i),$$

where $\sigma_i \in S_F$ and $\sum_{i=1}^{t} c_i = c$.

Let $\widetilde{\Gamma}$ be the subgroup of S_F generated by $\{\phi(a) \mid a \in A(G)\} \cup \{\sigma_i \mid 1 \leq i \leq t\}$ and for $\tau \in \widetilde{\Gamma}$, let $\mathbf{P}(\tau)$ denote the permutation matrix of τ defined in Eq. (1). We call the homomorphism $\rho : \widetilde{\Gamma} \to \mathrm{GL}_{\nu_F}(\mathbb{C})$ defined by $\tau \mapsto \mathbf{P}(\tau)$ the *permutation representation* of $\widetilde{\Gamma}$. Let $\rho_1 = \mathbf{1}, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of $\widetilde{\Gamma}$ with degree f_k for $1 \leq k \leq \ell$, so that $\sum_{k=1}^{\ell} f_k^2 = |\widetilde{\Gamma}|$. Then, the permutation representation ρ can be decomposed into a direct sum of irreducible representations: say $\rho = \bigoplus_{k=1}^{\ell} m_k\rho_k$, where m_k is the multiplicity of ρ_k. Moreover, there exists an invertible matrix $\mathbf{M}$ such that

$$\mathbf{M}^{-1}\rho(\tau)\mathbf{M} = \bigoplus_{k=1}^{\ell}(\rho_k(\tau) \otimes \mathbf{I}_{m_k})$$

for every $\tau \in \widetilde{\Gamma}$, where $\bigoplus_{k=1}^{s} \mathbf{A}_k = \mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_s$ denotes the direct sum of matrices with block diagonals $\mathbf{A}_1, \ldots, \mathbf{A}_s$ consecutively. Note that $\sum_{k=1}^{\ell} m_k f_k = \nu_F$. Furthermore, it is known [25] that $m_1 \geq 1$ since it is the number of orbits under the action of the group $\widetilde{\Gamma}$. Since (F, ω_F) is assumed to be (r, c)-regular, $\mathbf{D}_{(F,\omega_F)} = c\mathbf{I}_{\nu_F}$. Let $\widehat{\mathbf{M}} = \mathbf{I}_{\nu_G} \otimes \mathbf{M}$. Then we have

$$\begin{aligned}
&\widehat{\mathbf{M}}^{-1}\left[\mathbf{I}_{\nu_G\nu_F} - u\left(\sum_{\sigma\in\Gamma}\mathbf{A}(\sigma)\otimes\mathbf{P}(\sigma) + \mathbf{I}_{\nu_G}\otimes\mathbf{A}_{(F,\omega_F)}\right)\right.\\
&\quad\left. +\ u^2(\mathbf{Q}_{(G,\omega_G)}\otimes\mathbf{I}_{\nu_F} + \mathbf{I}_{\nu_G}\otimes\mathbf{D}_{(F,\omega_F)})\right]\widehat{\mathbf{M}}\\
&\quad = \widehat{\mathbf{M}}^{-1}\left[\mathbf{I}_{\nu_G\nu_F} - u\left(\sum_{\sigma\in\Gamma}\mathbf{A}(\sigma)\otimes\mathbf{P}(\sigma) + \mathbf{I}_{\nu_G}\otimes\sum_{i=1}^{t}c_i\mathbf{P}(\sigma_i)\right)\right.\\
&\qquad\left. +\ u^2(\mathbf{Q}_{(G,\omega_G)}\otimes\mathbf{I}_{\nu_F} + \mathbf{I}_{\nu_G}\otimes c\mathbf{I}_{\nu_F})\right]\widehat{\mathbf{M}}\\
&\quad = \bigoplus_{k=1}^{\ell}\left[\mathbf{I}_{\nu_G}\otimes\mathbf{I}_{f_k} - u\left(\sum_{\sigma\in\Gamma}\mathbf{A}(\sigma)\otimes\rho_k(\sigma) + \mathbf{I}_{\nu_G}\otimes\sum_{i=1}^{t}c_i\rho_k(\sigma_i)\right)\right.\\
&\qquad\left. +\ u^2((\mathbf{Q}_{(G,\omega_G)} + c\mathbf{I}_{\nu_G})\otimes\mathbf{I}_{f_k})\right]\otimes\mathbf{I}_{m_k}.
\end{aligned}$$

This proves the following theorem.

Theorem 6. *Let $(G \times^{\phi} F, \widetilde{\omega})$ be a weighted graph bundle with weighted base graph (G, ω_G) and weighted fibre graph (F, ω_F) which is (r, c)-regular with adjacency matrix $\mathbf{A}_{(F,\omega_F)} = \sum_{i=1}^{t} c_i\mathbf{P}(\sigma_i)$, $c_i \in \mathbb{C}$ and $\sigma_i \in S_F$. Let $\rho_1 = \mathbf{1}, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of the subgroup $\widetilde{\Gamma}$ of S_F generated by $\{\phi(a) \mid a \in A(G)\} \cup \{\sigma_i \mid 1 \leq i \leq t\}$ with degrees $f_1 = 1, f_2, \ldots, f_\ell$, respectively, and let the permutation representation $\rho : \widetilde{\Gamma} \to \mathrm{GL}_{\nu_F}(\mathbb{C})$ of $\widetilde{\Gamma}$ be decomposed into a direct sum of irreducible representations: say $\rho = \bigoplus_{k=1}^{\ell} m_k\rho_k$. Then the generalized Laplacian matrix $\mathbf{\Phi}_{(G\times^{\phi}F,\widetilde{\omega})}(u)$ of $(G \times^{\phi} F, \widetilde{\omega})$ is similar to the direct sum of block matrices*

$$\bigoplus_{k=1}^{\ell}(\Delta_k(u) \otimes \mathbf{I}_{m_k}),$$

where

$$\Delta_k(u) = \mathbf{I}_{\nu_G} \otimes \mathbf{I}_{f_k} - u \left(\sum_{\sigma \in \Gamma} \mathbf{A}(\sigma) \otimes \rho_k(\sigma) + \mathbf{I}_{\nu_G} \otimes \sum_{i=1}^{t} c_i \rho_k(\sigma_i) \right) + u^2((\mathbf{Q}_{(G,\omega_G)} + c\mathbf{I}_{\nu_G}) \otimes \mathbf{I}_{f_k})$$

and Γ is the subgroup of $\mathrm{Aut}(F)$ *generated by* $\{\phi(a) \mid a \in A(G)\}$. $\square$

If $\omega_G(e) = \omega_F(e') = 1$ for all $e \in E(G)$ and for all $e' \in E(F)$, then the formula in Theorem 6 gives the generalized Laplacian matrix of an (unweighted) graph bundle. Moreover, the adjacency matrix $\mathbf{A}_F$ of F can be written as (see p. 10 in [5])

$$\mathbf{A}_F = \mathbf{P}(\sigma_1) + \mathbf{P}(\sigma_2) + \cdots + \mathbf{P}(\sigma_r),$$

for some $\sigma_i \in S_F$. This gives the following corollary.

Corollary 7. *Let $G \times^{\phi} F$ be a graph bundle whose fibre graph F is r-regular with adjacency matrix $\mathbf{A}_F = \sum_{i=1}^{r} \mathbf{P}(\sigma_i)$, $\sigma_i \in S_F$. Let $\rho_1 = \mathbf{1}, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of the subgroup $\widetilde{\Gamma}$ of S_F generated by $\{\phi(a) \mid a \in A(G)\} \cup \{\sigma_i \mid 1 \le i \le r\}$ with degrees $f_1 = 1, f_2, \ldots, f_\ell$, respectively, and let the permutation representation $\rho : \widetilde{\Gamma} \to \mathrm{GL}_{\nu_F}(\mathbb{C})$ of $\widetilde{\Gamma}$ be decomposed into a direct sum of irreducible representations: say $\rho = \bigoplus_{k=1}^{\ell} m_k \rho_k$. Then the generalized Laplacian matrix $\mathbf{\Phi}_{G \times^{\phi} F}(u)$ of $G \times^{\phi} F$ is similar to the direct sum of block matrices*

$$\bigoplus_{k=1}^{\ell} (\Delta_k(u) \otimes \mathbf{I}_{m_k}),$$

where

$$\Delta_k(u) = \mathbf{I}_{\nu_G} \otimes \mathbf{I}_{f_k} - u \left(\sum_{\sigma \in \Gamma} \mathbf{A}(\sigma) \otimes \rho_k(\sigma) + \mathbf{I}_{\nu_G} \otimes \sum_{i=1}^{r} \rho_k(\sigma_i) \right) + u^2((\mathbf{Q}_G + r\mathbf{I}_{\nu_G}) \otimes \mathbf{I}_{f_k}),$$

and Γ is the subgroup of $\mathrm{Aut}(F)$ *generated by* $\{\phi(a) \mid a \in A(G)\}$. $\square$

Remark. Let the graph G and the r-regular graph F have the trivial weights $\omega_G = \omega_F = 1$ and let the subgroup Γ of $\mathrm{Aut}(F)$ generated by $\{\phi(a) \mid a \in A(G)\}$ be abelian. Then the generalized Laplacian matrix of the graph bundle $G \times^{\phi} F$ in Corollary 7 can be expressed in terms of the eigenvalues of $\mathbf{P}(\sigma)$ and $\mathbf{A}_F$. Note that every permutation matrix $\mathbf{P}(\sigma)$ commutes with the adjacency matrix $\mathbf{A}_F$ of F for all $\sigma \in \Gamma \le \mathrm{Aut}(F)$ (see [2]). Since Γ is assumed to be abelian, all representations ρ_k are linear and thus $\rho_k(\sigma)$ are eigenvalues of $\mathbf{P}(\sigma)$. Moreover, the matrices $\mathbf{A}_F$ and $\mathbf{P}(\sigma)$, $\sigma \in \Gamma$, are all diagonalizable and commute with each other, and hence there exists an invertible matrix $\mathbf{M}$ such that $\mathbf{M}^{-1}\mathbf{P}(\sigma)\mathbf{M}$ and $\mathbf{M}^{-1}\mathbf{A}_F\mathbf{M}$ are diagonal matrices for all $\sigma \in \Gamma$. Let $\lambda_1^{(\sigma)}, \ldots, \lambda_{\nu_F}^{(\sigma)}$ be the eigenvalues of the permutation matrix $\mathbf{P}(\sigma)$ and let $\mu_1, \ldots, \mu_{\nu_F}$ be the eigenvalues of the adjacency matrix $\mathbf{A}_F$ of F, so that

$$\mathbf{M}^{-1}\mathbf{P}(\sigma)\mathbf{M} = \begin{bmatrix} \lambda_1^{(\sigma)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_{\nu_F}^{(\sigma)} \end{bmatrix} \quad \text{and} \quad \mathbf{M}^{-1}\mathbf{A}_F\mathbf{M} = \begin{bmatrix} \mu_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mu_{\nu_F} \end{bmatrix}.$$

From this, one can see that the generalized Laplacian matrix $\mathbf{\Phi}_{G \times^\phi F}(u)$ is similar to

$$\bigoplus_{k=1}^{\nu_F} \left[\mathbf{I}_{\nu_G} - u \left(\sum_{\sigma \in \Gamma} \lambda_i^{(\sigma)} \mathbf{A}(\sigma) + \mu_i \mathbf{I}_{\nu_G} \right) + u^2 (\mathbf{Q}_{\nu_G} + r \mathbf{I}_{\nu_G}) \right].$$

3. Formulae for weighted complexities

In this section, we derive a decomposition formula for the (weighted) complexity of a (weighted) graph bundle $(G \times^\phi F, \widetilde{\omega})$ with regular fibre F, which is one of the main results in this paper. Let $\mathbf{A}_{(F,\omega_F)} = \sum_{i=1}^t c_i \mathbf{P}(\sigma_i)$ be the adjacency matrix of (F, ω_F) as a linear combination of permutation matrices as before. We also compute the complexity when the group $\widetilde{\Gamma}$ generated by $\{\phi(a) \mid a \in A(G)\} \cup \{\sigma_i \mid 1 \le i \le t\}$ is abelian.

We start by introducing two basic lemmas. Let (G, ω_G) be a weighted graph and let (F, ω_F) be a weighted (r, c)-regular graph. Recall that

$$\begin{aligned} \Delta_k(u) = \mathbf{I}_{\nu_G} \otimes \mathbf{I}_{f_k} - u \left(\sum_{\sigma \in \Gamma} \mathbf{A}(\sigma) \otimes \rho_k(\sigma) + \mathbf{I}_{\nu_G} \otimes \sum_{i=1}^t c_i \rho_k(\sigma_i) \right) \\ + u^2((\mathbf{Q}_{(G,\omega_G)} + c\mathbf{I}_{\nu_G}) \otimes \mathbf{I}_{f_k}). \end{aligned}$$

Now let $\delta_{(G,\omega_G)}(u) = \det \mathbf{\Phi}_{(G,\omega_G)}(u)$ and $\delta_k(u) = \det \Delta_k(u)$.

Lemma 8. $\delta_1(1) = 0$.

Proof. Note that, for all $\tau \in \widetilde{\Gamma}$, $\rho_1(\tau) = 1$ and $f_1 = 1$. Since $\sum_{\sigma \in \Gamma} \mathbf{A}(\sigma) = \mathbf{A}_{(G,\omega_G)}$, $\sum_{i=1}^t c_i = c$ and $\mathbf{Q}_{(G,\omega_G)} = \mathbf{D}_{(G,\omega_G)} - \mathbf{I}_{\nu_g}$, we have

$$\delta_1(u) = \det[(cu^2 - cu)\mathbf{I}_{\nu_G} + \mathbf{\Phi}_{(G,\omega_G)}(u)].$$

Noting $\mathbf{\Phi}_{(G,\omega_G)}(1) = \mathbf{L}_{(G,\omega_G)}$ the Laplacian matrix of (G, ω_G), we get $\delta_1(1) = \det \mathbf{\Phi}_{(G,\omega_G)}(1) = \det \mathbf{L}_{(G,\omega_G)} = 0$. □

Let $f(x, u) = \det[x\mathbf{I}_n - \mathbf{\Phi}_{(G,\omega_G)}(u)] = \sum_{i=0}^{\nu_G} a_i(u)\, x^i$ be the characteristic polynomial of $\mathbf{\Phi}_{(G,\omega_G)}(u)$. For a subset S of $\{1, \dots, \nu_G\}$, let $\mathbf{\Phi}_S(u)$ denote the matrix obtained by removing the rows and columns of $\mathbf{\Phi}_{(G,\omega_G)}(u)$ indexed by elements of S. It is known [11] that, for each $i = 0, \dots, \nu_G$,

$$a_i(u) = (-1)^{\nu_G - i} \sum_S \det \mathbf{\Phi}_S(u),$$

where the sum runs over all subsets S of $\{1, \dots, n\}$ with i elements. We observe that

(i) $a_0(1) = (-1)^{\nu_G} \det \mathbf{\Phi}_{(G,\omega_G)}(1) = (-1)^{\nu_G} \delta_1(1) = 0$ by Lemma 8, and
(ii) $a_1(1) = (-1)^{\nu_G - 1} \sum_{k=1}^{\nu_G} \det \mathbf{\Phi}_{\{k\}}(1) = (-1)^{\nu_G - 1} \cdot \nu_G \cdot \kappa(G, \omega_G)$.

Lemma 9. $\delta_1'(1) = c \cdot \nu_G \cdot \kappa(G, \omega_G) + \delta_{(G,\omega_G)}'(1)$.

Proof. The derivative of $\delta_1(u)$ is

$$\delta_1'(u) = \frac{\mathrm{d}}{\mathrm{d}u} \det \Big[(cu^2 - cu)\mathbf{I}_{\nu_G} + \mathbf{\Phi}_{(G,\omega_G)}(u) \Big] = (-1)^{\nu_G} \frac{\mathrm{d}}{\mathrm{d}u} f(cu - cu^2, u).$$

Thus we have

$$\begin{aligned}\delta_1'(1) &= (-1)^{\nu_G}\left[\frac{\mathrm{d}}{\mathrm{d}u} f(cu - cu^2, u)\right]_{u=1}\\ &= (-1)^{\nu_G}[a_1(u)(c - 2cu) + (-1)^{\nu_G}\delta'_{(G,\omega_G)}(u)]_{u=1}\\ &= (-1)^{\nu_G+1}ca_1(1) + \delta'_{(G,\omega_G)}(1)\\ &= c \cdot \nu_G \cdot \kappa(G, \omega_G) + \delta'_{(G,\omega_G)}(1). \quad \square\end{aligned}$$

Now we obtain a formula for the weighted complexity of $G \times^{\phi} F$ when F is (r, c)-regular.

Theorem 10. *Under the same hypothesis as those of Theorem 6, assume, in addition, that $G \times^{\phi} F$ is connected. Then*

$$\kappa(G \times^{\phi} F, \widetilde{\omega}) = \frac{1}{\nu_F}\kappa(G, \omega_G)\prod_{k=2}^{\ell}\delta_k(1)^{m_k},$$

where

$$\delta_k(1) = \det\Bigg[(\mathbf{D}_{(G,\omega_G)} + c\mathbf{I}_{\nu_G}) \otimes \mathbf{I}_{f_k} - \left(\sum_{\sigma\in\Gamma}\mathbf{A}(\sigma) \otimes \rho_k(\sigma) + \mathbf{I}_{\nu_G} \otimes \sum_{i=1}^{t} c_i\rho_k(\sigma_i)\right)\Bigg].$$

Proof. Note that the multiplicity m_1 of the irreducible representation $\rho_1 = \mathbf{1}$ is the number of orbits under the action of the group Γ. Thus, if $G \times^{\phi} F$ is connected, we have $m_1 = 1$. By Theorem 6, we have

$$\delta'_{(G\times^{\phi} F,\widetilde{\omega})}(u) = \delta_1'(u)\prod_{k=2}^{\ell}\delta_k(u)^{m_k} + \delta_1(u)\left[\prod_{k=2}^{\ell}\delta_k(u)^{m_k}\right]'.$$

And hence, by Lemma 8, we have

$$\delta'_{(G\times^{\phi} F,\widetilde{\omega})}(1) = \delta_1'(1)\prod_{k=2}^{\ell}\delta_k(1)^{m_k}.$$

By Theorem 2 and Lemma 9, it follows that

$$\begin{aligned}&2\left(\omega_+(G)\nu_F + \frac{c\nu_G\nu_F}{2} - \nu_G\nu_F\right) \cdot \kappa(G \times^{\phi} F, \widetilde{\omega})\\ &\quad = \{c \cdot \nu_G + 2(\omega_+(G) - \nu_G)\}\kappa(G, \omega_G) \cdot \prod_{k=2}^{\ell}\delta_k(1)^{m_k}.\end{aligned}$$

This completes the proof of the theorem. $\square$

As a special case, if the weight functions ω_G and ω_F are identically equal to the constant 1, we obtain the complexity of an (unweighted) graph bundle.

Corollary 11. *Under the same hypothesis as those of Corollary 7, we assume, in addition, that $G \times^{\phi} F$ is connected. Then*

$$\kappa(G \times^{\phi} F) = \frac{1}{\nu_F}\kappa(G)\prod_{k=2}^{\ell}\delta_k(1)^{m_k},$$

where

$$\delta_k(1) = \det\left[(\mathbf{D}_G + r\mathbf{I}_{\nu_G}) \otimes \mathbf{I}_{f_k} - \left(\sum_{\sigma\in\Gamma}\mathbf{A}(\sigma)\otimes\rho_k(\sigma) + \mathbf{I}_{\nu_G}\otimes\sum_{i=1}^{r}\rho_k(\sigma_i)\right)\right].$$

In particular, $\kappa(G)$ is a divisor of $\kappa(G \times^{\phi} F)$ for any F-bundle over G having a regular fibre F. $\square$

Without the assumption of the regularity of F in Corollary 11, the authors cannot determine whether $\kappa(G)$ is a divisor of $\kappa(G \times^{\phi} F)$ or not.

As a special case, let the group $\widetilde{\Gamma}$ generated by $\{\phi(a) \mid a \in A(G)\} \cup \{\sigma_i \mid 1 \le i \le t\}$ be abelian. In this case, the degree f_k of every irreducible representation ρ_k is equal to 1. Moreover, $\rho_k(\sigma)$ is an eigenvalue of $\mathbf{P}(\sigma)$ and $\sum_{i=1}^{t} c_i\rho_k(\sigma_i)$ is an eigenvalue of $\mathbf{A}_{(F,\omega_F)}$. Note that $\rho_1(\sigma) = \rho_1(\sigma_i) = 1$ for all σ and σ_i. Let $\lambda_k^{(\sigma)} = \rho_k(\sigma)$, $k = 1, \ldots, \nu_F$ (with multiplicities), be the eigenvalues of $\mathbf{P}(\sigma)$ and $\mu_k = \sum_{i=1}^{t} c_i\rho_k(\sigma_i)$ be the eigenvalues of $\mathbf{A}_{(F,\omega_F)}$. Then the weighted complexity of a graph bundle can be written as follows.

Corollary 12. *Let $(G \times^{\phi} F, \widetilde{\omega})$ be a weighted graph bundle with base graph (G, ω_G) and fibre graph (F, ω_F) which is (r, c)-regular with adjacency matrix $\mathbf{A}_{(F,\omega_F)} = \sum_{i=1}^{t} c_i\mathbf{P}(\sigma_i)$, $c_i \in \mathbb{C}$ and $\sigma_i \in S_F$. Assume that the group $\widetilde{\Gamma}$ generated by $\{\phi(a) \mid a \in A(G)\} \cup \{\sigma_i \mid 1 \le i \le t\}$ is abelian. Let $1 = \lambda_1^{(\sigma)}, \ldots, \lambda_{\nu_F}^{(\sigma)}$ be the eigenvalues of the permutation matrix $\mathbf{P}(\sigma)$ and let $c = \mu_1, \ldots, \mu_{\nu_F}$ be the eigenvalues of $\mathbf{A}_{(F,\omega_F)}$. Then*

$$\kappa(G \times^{\phi} F, \widetilde{\omega}) = \frac{1}{\nu_F}\kappa(G, \omega_G)\prod_{k=2}^{\nu_F}\det\left[\mathbf{D}_{(G,\omega_G)} + (c - \mu_k)\mathbf{I}_{\nu_G} - \sum_{\sigma\in\Gamma}\lambda_k^{(\sigma)}\mathbf{A}(\sigma)\right],$$

where Γ is the subgroup of Aut(F) *generated by $\{\phi(a) \mid a \in A(G)\}$.* $\square$

In unweighted case, we do not need the assumption that all elements in $\widetilde{\Gamma}$ commute with each other because the permutation matrix $\mathbf{P}(\sigma)$ commutes with the adjacency matrix $\mathbf{A}_F$ of F for each $\sigma \in \mathrm{Aut}(F)$ (see [2]). Thus $\mathbf{P}(\sigma)$ also commutes with $\mathbf{P}(\sigma_i)$ for each $i = 1, \ldots, r$. If the subgroup Γ of $\widetilde{\Gamma}$ generated by $\{\phi(a) \mid a \in A(G)\}$ is abelian, then $\widetilde{\Gamma}$ is also abelian. Hence we have the following corollary.

Corollary 13. *Let $G \times^{\phi} F$ be a graph bundle with an r-regular fibre F and let the group Γ generated by $\{\phi(a) \mid a \in A(G)\}$ be abelian. Let $\lambda_1^{(\sigma)}, \ldots, \lambda_{\nu_F}^{(\sigma)}$ be the eigenvalues of the permutation matrix $\mathbf{P}(\sigma)$, $\sigma \in \Gamma$ and let $\mu_1, \ldots, \mu_{\nu_F}$ be the eigenvalues of the graph F. Then*

$$\kappa(G \times^{\phi} F) = \frac{1}{\nu_F}\kappa(G)\prod_{k=2}^{\nu_F}\det\left[\mathbf{D}_G + (r - \mu_k)\mathbf{I}_{\nu_G} - \sum_{\sigma\in\Gamma}\lambda_k^{(\sigma)}\mathbf{A}(\sigma)\right]. \quad \square$$

4. Weighted complexities of graph coverings

In connection with the complexity of a graph covering, Mizuno and Sato [19] gave a formula for the complexity of a regular covering of a graph. Also they [20] extend it to the weighted complexity of a regular covering of a graph. We extend their works to those of any (regular or irregular) coverings.

Recall that an m-fold (regular or irregular) covering G^ϕ of a graph G is just a graph bundle $G \times^\phi F$ with the fibre $F = \overline{K}_m$. As a special case of Theorem 10, one can obtain the weighted complexity of a connected covering G^ϕ of G by substituting $c = c_i = 0$.

Theorem 14. *Let (G, ω_G) be a weighted graph and let G^ϕ be a connected m-fold (regular or irregular) covering of G. Let Γ be the subgroup of S_m generated by $\{\phi(a) \mid a \in A(G)\}$, and let $\rho_1 = 1, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of Γ, with degree f_k for each k, where $f_1 = 1$. Let $\rho : \Gamma \to \mathrm{GL}_m(\mathbb{C})$ be the permutation representation of Γ and m_k the multiplicity of ρ_k in ρ for each $k = 1, \ldots, \ell$. Then the weighted complexity of $(G^\phi, \widetilde{\omega})$ is*

$$\kappa(G^\phi, \widetilde{\omega}) = \frac{1}{m}\kappa(G, \omega_G) \prod_{k=2}^{\ell} \delta_k(1)^{m_k},$$

where $\delta_k(1) = \det\left[\mathbf{D}_{(G,\omega_G)} \otimes \mathbf{I}_{f_k} - \sum_{\sigma\in\Gamma} \mathbf{A}(\sigma) \otimes \rho_k(\sigma)\right]$. $\square$

In Theorem 14, if a connected covering G^ϕ is a regular covering of G, then there exists a subgroup $\mathcal{A}$ of Aut(G^ϕ) which acts regularly on G^ϕ and $G^\phi/\mathcal{A} \cong G$ (see [12]). Let $\rho_1 = 1, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of $\mathcal{A}$ with degree f_k for each k, where $f_1 = 1$. Then the multiplicity m_k is equal to the degree f_k of ρ_k and the fold number m is $|\mathcal{A}|$, the cardinality of $\mathcal{A}$. Thus we have the same formula as Theorem 5 in [20]:

$$\kappa(G^\phi, \widetilde{\omega}) = \frac{1}{|\mathcal{A}|}\kappa(G, \omega_G) \prod_{k=2}^{\ell} \delta_k(1)^{f_k},$$

where $\delta_k(1) = \det[\mathbf{D}_{(G,\omega_G)} \otimes \mathbf{I}_{f_k} - \sum_{\sigma\in\Gamma} \mathbf{A}(\sigma) \otimes \rho_k(\sigma)]$.

By taking trivial weights, one can also derive a formula for the complexity of an m-fold graph covering and the same formula as Theorem 1.6 in [19] for the complexity of a regular covering.

5. Complexities of the product of graphs

The cartesian product $G \times F$ of two graphs G and F has the vertex set $V(G \times F) = V(G) \times V(F)$, and vertices (u_1, v_1), (u_2, v_2) are adjacent whenever either $u_1 = u_2$ and $v_1v_2 \in E(F)$, or $u_1u_2 \in E(G)$ and $v_1 = v_2$. The cartesian product of graphs is an important and interesting operation on graphs. However, the complexity of the cartesian product of graphs has been explicitly evaluated for only a few cases (see [4,7,8,24]). One of them is $\kappa(P_n \times K_2)$, which is due to Sedláček [24]:

$$\kappa(P_n \times K_2) = \frac{1}{2\sqrt{3}}\left[(2+\sqrt{3})^n - (2-\sqrt{3})^n\right],$$

where P_n is the path of length n. Cvetković [8] gave an explicit formula for $\kappa(K_n^r) = \kappa(K_n \times \cdots \times K_n)$ as follows:

$$\kappa(K_n^r) = \kappa(K_n \times \cdots \times K_n) = n^{n^r - r - 1} \prod_{i=1}^{r} i^{\binom{n}{i}(n-i)^i}.$$

By using the Kel'mans and Chelnokov's formula and simple eigenvalue calculation, one can also get

$$\kappa(K_{n_1} \times \cdots \times K_{n_k}) = \frac{1}{\prod_{i=1}^{k} n_i} \prod_{\emptyset \neq S \subseteq [k]} N_S^{\prod_{i \in S}(n_i - 1)},$$

where $[k] = \{1, \ldots, k\}$ and $N_S = \sum_{i \in S} n_i$. The formulae for $\kappa(C_n \times K_2)$ and $\kappa(C_n \times C_m)$ were also given in [8], where C_n is the cycle of length n, but they were expressed as complicated trigonometric expressions. A remarkable paper on the complexity of the cartesian product of graphs is due to Boesch and Prodinger [4]. They used Chebyshev polynomials to give a formula for $\kappa(C_n \times K_m)$ as follows:

$$\kappa(C_n \times K_m) = \frac{n}{m} \left[\left(\frac{m + 2 + \sqrt{m^2 + 4m}}{2} \right)^n + \left(\frac{m + 2 - \sqrt{m^2 + 4m}}{2} \right)^n - 2 \right]^{m-1}.$$

They also derived the formulae for the complexities of the ladder $P_n \times K_2$ and the Möbius ladder M_n. Note that the Möbius ladder M_n is a graph bundle $C_n \times^{\phi} K_2$ (see Example 2). Recently, Hartsfield et al. [13] described a relation between the Laplacian polynomials of a graph G and $G \times Q_n$, where $Q_n = K_2 \times \cdots \times K_2$ is the n-cube.

In this section, as a consequence of our work, we give a formula for the complexity of the cartesian product of a graph and a regular graph. Recall that the cartesian product $G \times F$ is nothing but the graph bundle $G \times^{\phi} F$ with the trivial voltage assignment ϕ. Thus one can immediately get the following corollary of Theorem 10.

Corollary 15. *Let (G, ω_G) be a connected weighted graph and let (F, ω_F) be a connected (r, c)-regular weighted graph with $\mathbf{A}_{(F,\omega_F)} = \sum_{i=1}^{t} c_i \mathbf{P}(\sigma_i)$, $c_i \in \mathbb{C}$ and $\sigma_i \in S_F$. Let $\overline{\Gamma}$ be the subgroup of S_F generated by $\{\sigma_i \mid 1 \le i \le t\}$. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of $\overline{\Gamma}$, and f_k the degree of ρ_k for each k, where $f_1 = 1$. Let $\rho : \overline{\Gamma} \to \mathrm{GL}_{\nu_F}(\mathbb{C})$ be the permutation representation of $\overline{\Gamma}$ and m_k the multiplicity of ρ_k in ρ for each $k = 1, \ldots, \ell$. Then the weighted complexity of the cartesian product $(G \times F, \widetilde{\omega})$ of (G, ω_G) and (F, ω_F) is*

$$\kappa(G \times F, \widetilde{\omega}) = \frac{1}{\nu_F} \kappa(G, \omega_G) \prod_{k=2}^{\ell} \delta_k(1)^{m_k},$$

where

$$\delta_k(1) = \det\left[(\mathbf{D}_{(G,\omega_G)} + c\mathbf{I}_{\nu_G}) \otimes \mathbf{I}_{f_k} - \left(\mathbf{A}_{(G,\omega_G)} \otimes \mathbf{I}_{f_k} + \mathbf{I}_{\nu_G} \otimes \sum_{i=1}^{t} c_i \rho_k(\sigma_i)\right)\right].$$

□

By taking trivial weights and using Corollary 13, we have the formula for the complexity of the cartesian product of a graph and a regular graph.

Corollary 16. *Let G be a connected graph and let F be a connected r-regular graph. Let $r = \mu_1, \ldots, \mu_{\nu_F}$ be the eigenvalues of the adjacency matrix $\mathbf{A}_F$ of F. Then the complexity of the cartesian product $G \times F$ of G and F is*

$$\kappa(G \times F) = \frac{1}{\nu_F}\, \kappa(G) \prod_{k=2}^{\nu_F} \det[\mathbf{D}_G + (r - \mu_k)\mathbf{I}_{\nu_G} - \mathbf{A}_G]. \quad \square$$

The Laplacian eigenvalues of $G \times F$ consist of all numbers of the form $\lambda + \mu$, where λ and μ are Laplacian eigenvalues of G and F, respectively. The Kel'mans and Chelnokov formula asserts that, for connected graphs G and F,

$$\kappa(G \times F) = \kappa(G)\kappa(F) \prod_{\lambda \neq 0} \prod_{\mu \neq 0} (\lambda + \mu).$$

One can easily check that Corollary 16 also gives the same formula, when F is regular, since $\mathbf{D}_G - \mathbf{A}_G = \mathbf{L}_G$.

6. Some examples

In this final section, we give explicit formulae for the complexities of some special classes of graphs, that is, $P_n \times K_m$, $C_n \times K_m$, $C_n \times^{\phi} K_m$ and a 3-fold covering of K_4. Some formulae for the complexities of $P_n \times K_2$, $C_n \times K_m$ and $C_n \times^{\phi} K_2$ are already mentioned in Section 5. They used somewhat complicated calculations of determinants, eigenvalues or trigonometric equations. For the classes in our examples, however, we can avoid such a complication. The known formulae for $\kappa(P_n \times K_2)$, $\kappa(C_n \times K_m)$ and $\kappa(C_n \times^{\phi} K_2)$ can be simply derived from our result and the known formula for the determinant of a tridiagonal matrix. In addition, new formulae for $\kappa(P_n \times K_m)$ and $\kappa(C_n \times^{\phi} K_m)$ can be derived from the method.

Let

$$T_n(a,b,c) = \begin{bmatrix} a & b & & & & \mathbf{0} \\ c & a & b & & & \\ & c & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \\ \mathbf{0} & & & & c & a \end{bmatrix}_{n \times n}.$$

If $a^2 \neq 4bc$, the determinant of the tridiagonal matrix $T_n(a,b,c)$ is

$$\det T_n(a,b,c) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta},$$

where $\alpha = \frac{1}{2}(a + \sqrt{a^2 - 4bc})$, $\beta = \frac{1}{2}(a - \sqrt{a^2 - 4bc})$ (see Theorem 4.5 in [28]).

Example 1 (*The Complexity of the Product $G \times K_m$*). Let G be a graph and let K_m be the complete graph with m vertices. Note that K_m is $(m-1)$-regular with eigenvalues $\mu_1 = m-1$, $\mu_2 = \cdots = \mu_m = -1$. By Corollary 16, we have

$$\kappa(G \times K_m) = \frac{1}{m}\kappa(G) \det[\mathbf{D}_G + m\mathbf{I}_{\nu_G} - \mathbf{A}_G]^{m-1}.$$

If $G = P_n$ is the path with n vertices, then $\kappa(P_n) = 1$. Hence,

$$\begin{aligned}
\kappa(P_n \times K_m) &= \frac{1}{m} \det[\mathbf{D}_{P_n} + m\mathbf{I}_n - \mathbf{A}_{P_n}]^{m-1} \\
&= \frac{1}{m}[(m+1)^2 \det T_{n-2}(m+2,-1,-1) \\
&\quad - 2(m+1)\det T_{n-3}(m+2,-1,-1) + \det T_{n-4}(m+2,-1,-1)]^{m-1} \\
&= \frac{m^{m-2}}{(m^2+4m)^{\frac{m-1}{2}}} \left[\left(\frac{m+2+\sqrt{m^2+4m}}{2}\right)^n \right. \\
&\quad \left. - \left(\frac{m+2-\sqrt{m^2+4m}}{2}\right)^n\right]^{m-1}.
\end{aligned}$$

Here, the second equality is valid, as seen by taking the cofactor expansion of $\det[\mathbf{D}_{P_n} + m\mathbf{I}_n - \mathbf{A}_{P_n}]$ along the first row and again the expansion of each resulting factor along the last row.

As a special case, if $m = 2$, $L_n = P_n \times K_2$ is the ladder and its complexity is

$$\kappa(P_n \times K_2) = \frac{1}{2\sqrt{3}}\left[(2+\sqrt{3})^n - (2-\sqrt{3})^n\right],$$

as shown in [4] and [24].

Next, let $G = C_n$ be the cycle on n vertices. Then the complexity of $C_n \times K_m$ is

$$\begin{aligned}
\kappa(C_n \times K_m) &= \frac{n}{m}\det[2\mathbf{I}_n + m\mathbf{I}_n - \mathbf{A}_{C_n}]^{m-1} \\
&= \frac{n}{m}[(m+2)\det T_{n-1}(m+2,-1,-1) - 2\det T_{n-2}(m+2,-1,-1) - 2]^{m-1} \\
&= \frac{n}{m}\left[\left(\frac{m+2+\sqrt{m^2+4m}}{2}\right)^n + \left(\frac{m+2-\sqrt{m^2+4m}}{2}\right)^n - 2\right]^{m-1}.
\end{aligned}$$

If $m = 2$, $C_n \times K_2$ is the prism and its complexity is

$$\kappa(C_n \times K_2) = \frac{n}{2}\left[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2\right]$$

as shown in [4] and [24]. Also another formula for $\kappa(C_n \times K_2)$ was given in [7] and [8] using trigonometric expressions. □

Example 2 (*The Complexity of the Bundle $C_n \times^{\phi} K_m$*). Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Define a voltage assignment $\phi : A(C_n) \to S_m$ by $\phi(v_i v_{i+1}) = \sigma_1 = (1)$, for $1 \le i \le n-1$, and $\phi(v_n v_1) = \sigma_n = (1\ m)$. By using Corollary 11, we obtain the complexity of the bundle $C_n \times^{\phi} K_m$ as follows:

$$\begin{aligned}\kappa(C_n \times^{\phi} K_m) &= \frac{n}{m} \det[2\mathbf{I}_n + m\mathbf{I}_n - (\mathbf{A}(\sigma_1) - \mathbf{A}(\sigma_n))]^{m-1} \\ &= \frac{n}{m} \left[(m+2) \det T_{n-1}(m+2, -1, -1) - 2 \det T_{n-2}(m+2, -1, -1) + 2\right]^{m-1} \\ &= \frac{n}{m} \left[\left(\frac{m+2+\sqrt{m^2+4m}}{2}\right)^n + \left(\frac{m+2-\sqrt{m^2+4m}}{2}\right)^n + 2\right]^{m-1}.\end{aligned}$$

As a special case, if $m = 2$, $M_n = C_n \times^{\phi} K_2$ is the Möbius ladder and its complexity is

$$\kappa(M_n) = \frac{n}{2}\left[(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2\right],$$

as shown in [4]. This was also stated without proof in [2]. □

Example 3 (*The Weighted Complexity of a 3-fold Covering of K_4*). To compute the weighted complexity of a 3-fold covering of the complete graph K_4, let v_1, v_2, v_3, v_4 be the vertices of K_4, and let $\mathbf{A}_{(K_4,\omega)}$ be a symmetric weighted adjacency matrix of (K_4, ω), say

$$\mathbf{A}_{(K_4,\omega)} = \begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & d & 0 & f \\ c & e & f & 0 \end{bmatrix}.$$

Then, the weighted complexity of (K_4, ω) is

$$\begin{aligned}\kappa(K_4, \omega) = {} & abc + abe + abf + acd + acf + ade + adf + aef + bcd + bce \\ & + bde + bdf + bef + cde + cdf + cef.\end{aligned}$$

Let S_3 be the symmetric group on the set $\{1, 2, 3\}$. Define a permutation voltage assignment $\phi : A(K_4) \to S_3$ by

$$\begin{gathered}\phi(v_1, v_2) = \phi(v_1, v_3) = \phi(v_1, v_4) = \phi(v_2, v_4) = (1), \\ \phi(v_2, v_3) = (1\ 2) \qquad \text{and} \qquad \phi(v_3, v_4) = (1\ 2\ 3).\end{gathered}$$

One can see that the 3-fold covering K_4^{ϕ} is an irregular covering of K_4. Moreover, $\Gamma = \langle\{(1\ 2), (1\ 2\ 3)\}\rangle = S_3$, and S_3 has three irreducible representations $\rho_1 = \mathbf{1}$, ρ_2 (the sign representation), and ρ_3 which is given by

$$\rho_3((1)) = \mathbf{I}_2, \qquad \rho_3((1\ 2\ 3)) = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{bmatrix}, \qquad \rho_3((1\ 3\ 2)) = \begin{bmatrix} \zeta^2 & 0 \\ 0 & \zeta \end{bmatrix},$$

$$\rho_3((1\ 2)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \rho_3((2\ 3)) = \begin{bmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{bmatrix}, \qquad \rho_3((1\ 3)) = \begin{bmatrix} 0 & \zeta^2 \\ \zeta & 0 \end{bmatrix},$$

where $\zeta = \exp\frac{2\pi i}{3} = \frac{-1+\sqrt{-3}}{2}$. Note that their degrees f_1, f_2 and f_3 are 1, 1 and 2, respectively. Let $\rho : \Gamma \to \mathrm{GL}_3(\mathbb{C})$ be the permutation representation of Γ such that $\rho(\sigma) = \mathbf{P}(\sigma)$. Since $\rho = \mathbf{1} \oplus \rho_3$, we have the following from Theorem 14:

$$\begin{aligned}\kappa(K_4^{\phi}, \widetilde{\omega}) &= \frac{1}{3}\kappa(K_4, \omega)\ \det\left[\mathbf{D}_{(K_4,\omega)} \otimes \mathbf{I}_2 - \sum_{\sigma\in S_3} \mathbf{A}(\sigma) \otimes \rho_3(\sigma)\right] \\ &= \frac{1}{3}(abc + abe + abf + acd + acf + ade + adf + aef + bcd + bce + bde \\ &\quad + bdf + bef + cde + cdf + cef) \cdot \det(\mathbf{T}),\end{aligned}$$

where

$$\mathbf{T} = \begin{bmatrix} a+b+c & -a & -b & -c & 0 & 0 & 0 & 0 \\ -a & a+d+e & 0 & 0 & 0 & 0 & -d & 0 \\ -b & 0 & b+d+f & -\zeta f & 0 & -d & 0 & 0 \\ -c & 0 & -\zeta f & c+e+f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a+b+c & -a & -b & -c \\ 0 & 0 & -d & 0 & -a & a+d+e & 0 & 0 \\ 0 & -d & 0 & 0 & -b & 0 & b+d+f & -\zeta^2 f \\ 0 & 0 & 0 & 0 & -c & 0 & -\zeta^2 f & c+e+f \end{bmatrix}.$$

In particular, if $a = b = c = d = e = f = 1$, i.e., $\omega = 1$, then the complexity of K_4^{ϕ} is

$$\kappa(K_4^{\phi}) = 2787. \quad \square$$

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